A COMPREHENSIVE METHOD FOR EXOTIC OPTION PRICING

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Abstract. This work illustrates how several new pricing formulas for exotic options can be derived within a Lèvy framework by employing a unique pricing expression. Many existing pricing formulas of the traditional Gaussian model are obtained as a by-product.

1. Introduction

In the last decades the literature on option pricing under Lévy processes mushroomed. The reason is that Lévy processes have the flexibility to capture empirical characteristics in stock returns such as jumps in price due to market shocks and distributional properties such as skewness and semi-heavy tails, while maintaining most of the analytical tractability of the Gaussian model. In this paper we adopt the class of regular Lévy processes of exponential type (RLPE) as the driving processes, following [7]. As [7] points out, it is the most tractable subclass of Lévy process from the analytical point of view if the Brownian motion is not available. Their characteristic exponents $\psi(\xi)$ enjoy very favourable properties as symbols of pseudo differential operators, since their real part behaves as $c\,|\xi|^{\upsilon}$ as $|\xi| \to \infty$ in the strip of regularity, with positive c and υ. Thus the integrals appearing in the pricing formulas are absolutely convergent thanks to the terms of the form $e^{-\tau\psi(\xi)}$. Moreover, one can differentiate under the integral sign or shift the line of integration by using the Cauchy theorem for holomorphic functions. Such properties allow for a great flexibility of the method when working out the analytical pricing formulas for several exotic options. Moreover, such a class incorporates most of the models which have been proposed as an alternative to the traditional geometric Brownian motion, including the Normal Inverse Gaussian (NIG) (Barndorff-Nielsen [6]), the Hyperbolic (H) (Eberlein and Keller [13]) and the more general Generalized Hyperbolic (GH) (Eberlein and Prause [15]), the four-parameter distribution named CGMY after the names of Carr, Geman, Madan and Yor [10] and generalized in [11], just to quote the most popular ones. The development of pricing models replacing the traditional underlying source of randomness, the Brownian motion, by a Lévy process has fostered a good deal of work on exotic option pricing which parallels the existing results in the Gaussian framework. (See [27], for a compendium of recent research on the topic).

In this article the focus is on the pricing of European exotics and the aim is to present a valuation formula which is the most comprehensive as possible, in that

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several types of options can be priced directly and no specific method has to be devised for each of them. The formula is tailored to valuate discretely monitored options, which are the most popular ones in view of the regulatory issues and the trading practice. However, the continuous counterpart can be derived in some cases (see Example 2). The method employed in this work is to start from (multi-period) digital options as building-blocks and then to price the more complex options in terms of such elementary contracts. The idea that a broad class of financial derivatives can be evaluated in terms of elementary contracts such as digital options has been applied in the traditional Gaussian framework (see [9] and [23]) but it has not yet fully exploited in a non-Gaussian modeling. By providing the non-Gaussian counterpart of this view, we are able to obtain new pricing formulas in the Lèvy environment and to throw a new insight into some known pricing expressions. This method has been anticipated in [3]. This work employs a more general setting and provides further examples of exotic option prices. Since a Brownian motion is a RLPE of order 2 and any exponential type - and thus is captured in this framework - each example includes the known pricing expressions of the classical Gaussian modeling. An emphasis is put on deriving the existing formulas of the traditional Black-Scholes modeling from our more general framework, so that the reader is confronted with the more familiar expressions. To our surprise, the Lemma which is employed to the purpose is not found in the Lévy literature and therefore a proof is given. Some applications of the main result are given in Section 4. They serve as an illustration of the flexibility of our valuation formula and integrate the several examples developed by the author in a previous work [3].

This paper is organized as follows. Section 2 outlines the main definitions and notation concerning Lévy processes and provides a Lemma linking the Gaussian multivariate distribution to the Fourier transform-based Lèvy setting. This Lemma will be used extensively throughout the paper and has a *per se* mathematical interest. Section 3 presents the general valuation formula, while some examples of exotic option prices are provided in Section 4. The choice of the examples has been dictated by the wish to provide new valuation formulas and presenting them in a general form (for example, N-fold compound options instead of just 2-fold ones, flexible Asian options, both call and put options in one expression, etc.). However they are by no means exhaustive with respect to the potentialities of the approach.

2. Preliminaries

Let us consider a Lévy market, i.e. a model of financial market with a deterministic saving account e^{rt}, $r \geq 0$, and a stock following a stochastic process $S_t = e^{X_t}$, where $(X_t)_{t\geq 0}$ is a Lévy process. As usual in a Lévy setting, the Lévy process replaces the Brownian motion which is employed in the classical modeling of stock prices. Here we assume that the stock price S_t is e^{X_t}, where $(X_t)_{t\geq 0}$ is a one-dimensional RLPE, i.e. a regular Lévy process of order $\upsilon \in]0,2]$ and exponential type $[\lambda_-, \lambda_+]$, $\lambda_- < 0 < \lambda_+$. This means that $(X_t)_{t\geq 0}$ has a characteristic exponent $\psi(\xi)$ which admits a representation of the form:

$$\psi(\xi) = -i\mu\xi + \phi(\xi) \tag{2.1}$$

where ϕ is holomorphic in the strip $\operatorname{Im}\xi \in]\lambda_-, \lambda_+[$, continuous up to the boundary

of the strip, and $\phi(\xi) = C\left|\xi\right|^{\upsilon} + O(\left|\xi\right|^{\upsilon_1})$ for $|\xi| \longrightarrow \infty$ and $\left|\phi'(\xi)\right| \leq C(1 + \left|\xi\right|^{\upsilon_2})$ in $\operatorname{Im}\xi \in [\lambda_-, \lambda_+]$ with $\upsilon_1, \upsilon_2 < \upsilon$. (See [B-L] for a comprehensive and Finance-oriented theory of RLPE processes). In order to price contingent claims on the stock, it is convenient to consider an equivalent martingale measure (EMM) Q which makes the discounted price process $e^{-rt}S_t$ a martingale. Let ψ_P (respectively ψ_Q) denote the characteristic exponent with respect to the historic measure P (an EMM Q, respectively), i.e. $E_P(e^{i\xi X_t}) = e^{-t\psi_P(\xi)}$ $(E_Q(e^{i\xi X_t}) = e^{-t\psi_Q(\xi)}$, respectively). We assume that the discounted stock price $e^{-rt}S_t$ is a martingale under Q, that is, we assume that the equivalent martingale measure condition (EMM-condition) $r + \psi_Q(-i) = 0$ holds. Furthermore, the additional condition $\lambda_- < -1$, which is usually assumed to let $-i$ belong to the strip of regularity of ψ, is supposed to hold true. In the sequel further restrictions on $\lambda_\pm$ will be posed, depending on the kind of exotic option under study. We recall that Q is not unique in a general Lévy setting. We do not dive into the problem of choosing a martingale measure and refer to the well-established literature on Lévy processes in Finance dealing with this topic extensively (see [17], for example). A popular method is based on Esscher transform and the expression for $\psi_Q(\xi)$ is obtained by first solving the equation $\psi_P(-ih) - \psi_P(-ih-i) = r$ for h, and then letting $\psi_Q(\xi) = \psi_P(\xi - ih) - \psi_P(-ih)$. In what follows we assume that an EMM Q is chosen so that X_t is a RLPE under it and we will omit the subscript Q both in E and ψ.

Let $g(X_T)$ denote the terminal payoff of an option on S_t at the expiry date T. Then the no-arbitrage price of the option at the current time t $(t < T)$ is given by:

$$F(S_t, t) = E[e^{-r(T-t)}g(X_T) \mid X_t = \ln(S_t)] \tag{2.2}$$

An explicit expression for $F(S_t, t)$ can be obtained in terms of the Fourier transform $\widehat{g}(\xi)$ in the complex plane. Indeed, if $e^{\omega x}g(x) \in L^1(\mathbb{R})$ for some $\omega \in]\lambda_-, \lambda_+[$, then the Fourier transform $\widehat{g}(\xi)$ of g can be defined on $\operatorname{Im}\xi = \omega$ and one obtains:

$$F(S_t, t) = \frac{1}{2\pi}\int_{-\infty+i\omega}^{+\infty+i\omega} e^{i\xi\ln S_t - (T-t)(r+\psi(\xi))}\widehat{g}(\xi)d\xi \tag{2.3}$$

In the following sections this set-up is extended to the more general case of path-dependent options where the payoff g depends on a set of fixed asset price monitoring times, $T_1 < ... < T_M \leq T$, so that several types of exotic options can be priced in this framework.

Finally we recall that a Brownian motion is a RLPE of order 2 and any exponential type. Therefore the pricing formulas of the traditional Gaussian approach can be obtained in our framework for several exotics. To the purpose we need the following:

Lemma 1. *Let $C = (c_{kj})$ denote an $N \times N$ correlation matrix. Then for any $\omega_k > 0$, for any real number d_k and with $w_k = \pm 1$, $k = 1, ...N$, the following identity holds:*

$$\frac{1}{(2\pi i)^N}\int_{-\infty-iw_N\omega_N}^{+\infty-iw_N\omega_N} \cdots \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{\Sigma_{k=1}^N i\xi_k d_k - \frac{1}{2}\Sigma_{k,j=1}^N c_{kj}\xi_k\xi_j} \frac{1}{\Pi_{k=1}^N \xi_k} d\xi_1 ... d\xi_N =$$
$$= [\prod_{k=1,\dots,N} w_k] N_N(w_1 d_1, ... w_N d_N; WCW)$$

where N_N denotes the N-variate multinormal cumulative distribution function and W is the $N \times N$ diagonal matrix with entries w_k, $k = 1, ...N$.

Proof. Consider the case $w_k = 1$, $k = 1, ...N$, at first. Let $\Omega = (\omega_1, ..., \omega_N)$ and let $\varkappa_\Omega(x_1, ..., x_N)$ denote $\exp(-\sum\limits_{j=1,...,N} \omega_j x_j) I_{[-d_1,+\infty)}(x_1)...I_{[-d_N,+\infty)}(x_N)$ which belongs to $L^1(\mathbb{R}^N)$. Its Fourier transform in $(\xi_1^R, .., \xi_N^R) \in \mathbb{R}^N$ is:
$\exp[i \sum\limits_{j=1,...,N} d_j \xi_j] / \prod\limits_{j=1,...,N} i\xi_j$, with $\xi_j = \xi_j^R - i\omega_j$. On the other hand, $\exp(-\frac{1}{2}\Xi^T C \Xi)$ is the Fourier transform of $\phi(X) = \frac{1}{(2\pi)^{M/2}\sqrt{\det C}} \exp(-\frac{1}{2} X^T C^{-1} X)$. Thus the left-hand side term can be written as a convolution of $\varkappa_\Omega$ and ϕ, that is, as
$\frac{1}{(2\pi)^{M/2}\sqrt{\det C}} \int_{\mathbb{R}^N} \varkappa_\Omega(-Y) e^{-\Omega^T Y} \exp(-\frac{1}{2} Y^T C^{-1} Y) dY$,
which is $\int_{-\infty}^{d_1} ... \int_{-\infty}^{d_N} \frac{\exp(-\frac{1}{2} Y^T C^{-1} Y)}{(2\pi)^{M/2}\sqrt{\det C}} dy_1...dy_N = N_N(d_1, ...d_N; C)$. The general case follows changing to variables $\xi_j' = w_j \xi_j$ in the integrals of the left-hand term and replacing each d_k with $\widehat{d}_k = w_k h_k$ and C with WCW in the previous argument.

3. Set-up and main result

In this section an almost universal option pricing formula is proved within the Lévy framework described in the previous section. In particular we derive the arbitrage-free price for generalized multi-period exotic power digital options. Such options are the building blocks for a broad class of exotic options with a single underlying asset, because several exotic options can be expressed as static portfolios of these multi-period power digitals. The idea of pricing several exotic options by a single universal formula is due to [9], where the classical Gaussian case was studied. Thus this Section provides the generalization to the Lévy framework and the resulting formula is a most comprehensive one, both in terms of the kind of option and of the stochastic process driving the underlying asset. Since the aim is to establish a unifying set-up, the formal notation is a bit involved and is laid down along the lines of [9].

Assume that the payoff of an option depends on M fixed asset price monitoring times, $T_1 < ... < T_M$, where, for simplicity's sake, $T_M = T$, the expiry date of the option. Let $\mathbf{T}$ denote the set of times $[t, T_1, ..., T_M]$, where t is the current time, $t < T_1$. Let S_k denote the price of the underlying asset at the monitoring time T_k and let $\mathbf{S}$ denote the M-dimensional vector assembling all the components S_k which are relevant for the option under study. If γ is M-dimensional , then $\mathbf{S}^\gamma$ denotes $S_1^{\gamma_1}...S_M^{\gamma_M}$ and γ is referred to as the payoff index vector. We assume that $S_t = e^{X_t}$, where X_t is a Lévy process. Thus, in most cases, it will be convenient to work directly with X_k, the value of X at T_k, and with the vector $\mathbf{X}= (X_1, ..., X_M)$. Since we want to treat call and put options together, we introduce W, a diagonal matrix with all the diagonal entries w_{ii} equal to ± 1. To simplify notation, w_{ii} will be denoted by w_i. Then both the indicator functions $I_{[K_i,+\infty)}(X_i)$ and $I_{(-\infty,K_i]}(X_i)$ may be encompassed in a unique notation, $\mathbf{1}_1(w_i X_i \geq w_i K_i)$, depending on the sign of w_i. Let $\mathbf{K}$ denote the exercise price vector (of dimension N) and let $\mathbf{1}_N(\mathbf{Y} \geq \mathbf{K})$ denote the $N-$dimensional indicator function $\prod\limits_{i=1}^{N} \mathbf{1}_1(Y_i \geq K_i)$. In order to give a greater flexibility to the approach, the exercise condition matrix is introduced. Such term will denote any $N \times M$ matrix $A = (a_{nk})$ involved in the payoff, where N is the exercise dimension. For example, the payoff of discrete mean Asian options depends on $\sqrt[M]{\Pi_{k=1,...,M}\ S_k} \lesseqgtr \widetilde{K}$ or, equivalently, $\sum\limits_{k=1,...M} \frac{1}{M} X_k \lesseqgtr \ln \widetilde{K} = K$. Thus

their payoff can be expressed in terms of $\mathbf{1}_1(w_1 A\mathbf{X}\geq w_1 K)$, where $A=[\frac{1}{M},...,\frac{1}{M}]$ is a $1\times M$ matrix. On the other hand, the payoff of a discretely monitored digital option $\mathbf{1}_M(WX\geq WK)$ can be expressed in the general form $\mathbf{1}_M(WA\mathbf{X}\geq W\mathbf{K})$ by taking A as the $M\times M$ identity matrix. All the useful parameters are summarized in the payoff parameter set $\mathbf{P}=[(\gamma_1,...\gamma_M),\mathbf{K},W,A]$.

Let $F(S_t,t;\mathbf{T},\mathbf{P})$ denote the value of a multi-period binary whose payoff is specified in terms of $\mathbf{T}$ and $\mathbf{P}$ and where S_t denotes the value of the underlying asset at the current time t. More specifically, the expiry T payoff function is $\exp[\sum\limits_{k=1}^{M}\gamma_k X_k].\mathbf{1}_N(WA\mathbf{X}\geq W\mathbf{K})$. In most applications $\gamma_k=0$ for $k\neq M$. The following Proposition gives the arbitrage-free current value of this generalized multi-period option under the assumption that the value S_t of the underlying asset S_t is e^{X_t}, where X_t is a RLPE of order $\upsilon\in]0,2]$ and exponential type $[\lambda_-,\lambda_+]$. The main result of the paper is the following:

Proposition 1. *The following valuation formula holds:*

$$F(S_t,t;\boldsymbol{T},\boldsymbol{P})=\frac{e^{-r(T_M-t)}}{(2\pi i)^N}S_t^{\sum\limits_{k=1}^{M}\gamma_k}\prod_{n=1}^{N}w_n\int_{-\infty-iw_N\omega_N}^{+\infty-iw_N\omega_N}...\int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1}\frac{1}{\prod\limits_{n=1,...N}\xi_n}$$

$$\exp[i\sum_{n=1}^{N}\xi_n(\sum_{k=1}^{M}a_{nk}\ln S_t-K_n)-\Psi(t,\xi_1,...\xi_N)]d\xi_1..d\xi_N$$

where $\Psi(t,\xi_1,...\xi_N)=\sum\limits_{j=1}^{M}(T_j-T_{j-1})\psi(\sum\limits_{n=1}^{N}\sum\limits_{k=j}^{M}a_{nk}\xi_n-i\sum\limits_{k=j}^{M}\gamma_k)$ *with* $T_0=t$, $\omega_n>0$ *for* $n=1,...N$ *and* $\sum\limits_{k=j}^{M}\left[\sum\limits_{n=1}^{N}w_n\omega_n a_{nk}+\gamma_k\right]\in]-\lambda_+,-\lambda_-[$ *for* $j=1,...,M.$

Before proving Proposition 1 we give a pricing formula for the simple case of a power digital option, which is slightly more general than Proposition 1 in [3].

Lemma 2. *Let $F(S_t,t)$ denote the current arbitrage-free price of the power option with expiry T payoff function $S_T^{\gamma}\mathbf{1}_1(waX_T\geq wK)$, $a\neq 0$, $w=\pm 1$, $X_t=\ln S_t$. Then for any $\omega>0$ such that $aw\omega+\gamma\in]-\lambda_+,-\lambda_-[$, one can write*

$$F(S_t,t)=\frac{wS_t^{\gamma}}{2\pi i}\int_{-\infty-iw\omega}^{+\infty-iw\omega}e^{i\xi[a\ln S_t-K]-(T-t)[r+\psi(a\xi-i\gamma)]}\frac{1}{\xi}d\xi \tag{3.1}$$

Proof. The Fourier transform of the payoff function $g(X_T)=e^{\gamma X}\ \mathbf{1}_1(waX_T\geq wK)$ is $\widehat{g}(\eta)=\frac{w.sgn(a)}{i(\eta+i\gamma)}\exp[(\gamma-i\eta)\frac{K}{a}]$ with $\operatorname{Im}\eta=-w\omega_\gamma$ for any ω_γ such that $\gamma-w\omega_\gamma\gtrless 0$ whenever $aw\lessgtr 0$ and $w\omega_\gamma\in]-\lambda_+,-\lambda_-[$. Then:

$F(S_t,t)=\frac{w.sgn(a)}{2\pi i}\int_{-\infty-iw\omega_\gamma}^{+\infty-iw\omega_\gamma}e^{i\eta[\ln(S_t)-\frac{K}{a}]-(T-t)(r+\psi(\eta))}\frac{1}{\eta+i\gamma}d\eta$

which gives (3.1) changing to variables $\eta+i\gamma=a\xi$ and letting $\omega_\gamma-w\gamma=a\omega$.

Remark 1. The expression (3.1) can be rewritten in terms of pseudo differential operators as follows:

$F(S_t,t)=wS_t^{\gamma}\exp[-(T-t)(r+\psi(aD_x-i\gamma))]I^{(w)}(\ln\frac{S_t}{K})$

where $I^{(w)}$ denotes the indicator function $I^{(w)}(x)=I_{[0,+\infty)}(wx)$. Here the notation $P(D_x)$, with $D_x=-i\partial_x$, denotes a pseudo differential operator whose symbol is

$P(\xi)$. Alternatively, $F(S_t,t) = f(X_t,t)$ can be viewed as a solution to the pseudo differential equation: $[\partial_t - (r+\psi(D_x))]f(X_t,t) = 0$ with the final condition $f(X_T,T) = e^{\gamma X_T}\mathbf{1}_1(waX_T \geq wK)$.

Lemma 3. *Let $F(S_t,t)$ denote the current arbitrage-free price of the power option with expiry T payoff function $S_T^\gamma \mathbf{1}_N(w_n a_n X_T \geq w_n K_n; n=1,...,N)$, $a_n \neq 0$, $w_n = \pm 1$, $X_t = \ln S_t$. Then, for any $\omega_n > 0$ such that $\sum\limits_{n=1}^{N} w_n\omega_n a_{nk} + \gamma \in]-\lambda_+, -\lambda_-[$, one can write*

$$(3.2)\qquad F(S_t,t) = \frac{e^{-r(T-t)}}{(2\pi i)^N} S_t^\gamma \prod_{n=1}^{N} w_n \int_{-\infty - iw_N\omega_N}^{+\infty - iw_N\omega_N} ... \int_{-\infty - iw_1\omega_1}^{+\infty - iw_1\omega_1} \frac{1}{\prod\limits_{n=1,...N} \xi_n}.$$

$$\exp[i\sum_{n=1}^{N}\xi_n(a_nX_t - K_n) - \Psi(t,\xi_1,...\xi_N)]d\xi_1..d\xi_N$$

where $\Psi(t,\xi_1,...\xi_N) = (T-t)\psi(\sum\limits_{n=1}^{N} a_n\xi_n - i\gamma)$

Proof. The result follows by writing the Fourier transform of the payoff as a convolution of N terms and by arguing as in Lemma 1.

Proof of Proposition 1. Let us first prove the case $N=1$.
Let $\mathbf{P}= [(\gamma_1,...\gamma_M), K, w, (a_1,...a_M)]$ with $w = \pm 1$. For $m = 1,...,M$ let $K_m^* = K - \sum_{k=1}^{m-1} a_k X_k$, $\gamma_m^* = \sum_{k=1}^{m-1}\gamma_k X_k$ ($K_1^* = K$, $\gamma_m^* = 0$). Let $f_m(X_t,t)$ solve $\partial_t f_m - (r+\psi(D_x))f_m = 0$ for $t \in [T_{m-1}, T_m]$, with $f_m(T_m, X_m) = f_{m+1}(T_m, X_m)$ for $m < M$, and $f_M(T_M, X_M) = e^{\gamma_M X_M + \gamma_M^*}\mathbf{1}_1(wa_M X_M \geq wK_M^*)$. In view of Lemma 2 and Remark 1 one has:
$f_M(X_t,t) = \frac{we^{\gamma_M X_t + \gamma_M^*}}{2\pi i}\int_{-\infty - iw\omega}^{+\infty - iw\omega} e^{i\xi[a_M X_t - K_M^*] - (T_M - t)[r+\psi(a_M\xi - i\gamma_M)]}\frac{1}{\xi}d\xi.$
Then one can prove recursively that for any m:

$$f_m(X_t,t) = \frac{we^{-r(T_M-t)}\exp(\gamma_h^* + \sum\limits_{k=m}^{M}\gamma_k)}{2\pi i}\int_{-\infty - iw\omega}^{+\infty - iw\omega}\exp[i\xi(\sum_{k=m}^{M} a_k X_t - K_m^*) - \Psi_m(t,\xi)]\frac{1}{\xi}d\xi$$

where $\Psi_m(t,\xi) = \sum\limits_{j=m}^{M}(T_j - T_{j-1})\psi(\xi\sum\limits_{k=j}^{M} a_k - i\sum\limits_{k=j}^{M}\gamma_k)$ with $T_{m-1} = t$, $\omega > 0$ and $\sum\limits_{k=j}^{M}[w\omega a_k + \gamma_k] \in]-\lambda_+, -\lambda_-[$ for $j = m,...,M$. Thus $h = 1$ yields the result in the case $N = 1$. Finally, the general case is proved along the same lines, by employing Lemma 3 and arguing recursively.

In the sequel the pricing formulas for some exotics are obtained as an application of Proposition 1. Note that our pricing formula are new in the Lévy environment. Moreover, our approach casts a new light also on the Gaussian case that is obtained as a by-product.

4. Examples

In this section we provide some examples to illustrate the flexibility of the main formula. Most of the examples we give are new in the Lévy framework. Note that we have chosen not to include in this section the textbook examples and the most elementary examples concerning popular options: some are given in [3]. A few lines in each example are devoted to show how the well-known pricing expressions of the traditional Gaussian setting can be obtained as special cases of ours quite straightforwardly. Finally note that the references we give here are not exhaustive, because the focus of this paper is on the analytical formulas, while the amount of work on the purely computational issues is generally omitted.

1) *Forward-start options.* Forward-start options are options that are only effective at some pre-specified time after they have been bought or sold. They are the building-blocks of more complex options. For example, a cliquet option is a series of consecutive forward-start options, where each option becomes active when the former expires. A valuation of forward-start options is given in [29] in the classical Gaussian setting. Herewith the generalization to a Lévy setting is presented.

Let T be the expiration time and let $T_1 < T$ be the time at which the option starts. The strike price is set to be the underlying asset price at T_1. Thus the final payoff is $\max\{w(S_{T_2} - S_{T_1}), 0\}$ with $w = \pm 1$.

As an application of Proposition 1 we give the current value of the forward-start option $F(S_t, t)$ in the above-mentioned Lèvy setting, under the assumptions $\lambda_+ > 0$ and $\lambda_- < -1$.

$F(S_t,t) = \frac{e^{-r(T_2-t)}}{2\pi i} S_t (\int_{-\infty - iw\omega_2}^{+\infty - iw\omega_2} \frac{1}{\xi} \exp[-(T_2 - T_1)\psi(\xi - i)] d\xi$
$- \int_{-\infty - iw\omega_1}^{+\infty - iw\omega_1} \frac{1}{\xi} \exp[-(T_2 - T_1)\psi(\xi)] d\xi)$

where $\omega_1 \in]0, -\lambda_-[$, $\omega_2 \in]0, -\lambda_- - 1[$ if $w = 1$ and $\omega_1 \in]0, \lambda_+[$, $\omega_2 \in]0, \lambda_+ + 1[$ if $w = -1$.

Note that the greeks Δ (which is equal to $F(S_t,t)/S_t$) and $\Gamma = 0$ do not depend on S_t. In the Gaussian case, $\psi(\xi) = i(\frac{\sigma^2}{2} - r)\xi + \frac{\sigma^2}{2}\xi^2$, the formula above becomes:

$\frac{1}{2\pi i} S_t (\int_{-\infty - iw\omega_2}^{+\infty - iw\omega_2} \frac{1}{\xi} \exp[i(r + \frac{\sigma^2}{2})(T_2 - T_1)\xi - \frac{(T_2 - T_1)\sigma^2}{2}\xi^2] d\xi$
$- e^{-r(T_2 - T_1)} \int_{-\infty - iw\omega_1}^{+\infty - iw\omega_1} \frac{1}{\xi} \exp[i(r - \frac{\sigma^2}{2})(T_2 - T_1)\xi - \frac{(T_2 - T_1)\sigma^2}{2}\xi^2] d\xi)$

which is $wS_t[N(w(\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{T_2 - T_1}) - e^{-r(T_2 - T_1)} N(w(\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T_2 - T_1})$, i.e. the formula found in [29].

2) Asian options. Asian options under Lèvy processes have been priced in a number of papers (see [5], [18], [21]). In this subsection we show how a pricing formula for geometric Asian options is easily obtained from our general result. At first we consider discrete Asian options - whose payoff depends on a discrete average of the asset price at N monitoring times, $T_1 < ... < T_M$. The continuous average case is obtained as a limit. Consider a forward-start fixed strike Asian option with strike price K. Note that a pricing expression for floating Asian options is easily obtained in view of the symmetry relationship proved in [14]. Let $T = T_M$ be the maturity date and let $T' = T_1$ be the time at which the averaging starts. The payoff is $\max(w[\Sigma_M - K], 0)$, where $\Sigma_M = (\prod_{j=1,...,M} S_{T_j})^{\frac{1}{M}}$ and w is the binary indicator.

In terms of $X_t = \ln(S_t)$ the payoff is:

$w \prod_{j=1,...,M} e^{\frac{1}{M} X_{T_j}} \mathbf{1}_1(wA\mathbf{X} \geq w\ln(K)) - wK\mathbf{1}_1(wA\mathbf{X} \geq w\ln(K))$, with $A = [\frac{1}{M}, ..., \frac{1}{M}]$.

Thus Proposition 1 applies with $N = 1$ and yields the following valuation formula, after some algebraic manipulation:

$$F(S_t,t) = -\frac{Ke^{-r(T_M-t)}}{2\pi}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\frac{1}{\xi(\xi+i)}\exp[i\xi\ln\frac{S_t}{K} - \sum_{j=1}^{M}(T_j - T_{j-1})\psi(\xi\frac{M-j}{M})]d\xi$$

with $T_0 = t$, $\omega \in]1, -\lambda_-[$ $(]0, \lambda_+[)$ if $w = 1$ $(w = -1)$. Note that the analogous expression obtained in [18], (10), is derived throughout a different argument, that is considering the distribution of $\ln(\Sigma_M)$.

The pricing formula for the continuous-time monitoring case, where the geometric average is $\exp[\frac{1}{T-T'}\int_{T'}^{T}\ln(S_t)dt]$, follows from the discrete pricing formula just letting $M \to \infty$. Note that the limit can be computed under the integral sign in view of the nice behavior of ψ. In particular, for the continuous case, one has:

$$F(S_t,t) = -\frac{Ke^{-r(T-t)}}{2\pi}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\frac{1}{\xi(\xi+i)}\exp[i\xi\ln\frac{S_t}{K} - \int_0^1\psi(\xi(1-y))dy]d\xi.$$

Let us now see that our formula collapses to the know valuation formula for discretely monitored Asian options in the Gaussian case (see [29]). Let h denote the averaging frequency, that is, $T_j = T - (M-j)h$, $j = 1, ..., M$. Then, in the Gaussian case, $\sum_{j=1}^{M}(T_j - T_{j-1})\psi(\xi\frac{M-j}{M}) = -ih(r - \frac{\sigma^2}{2})\xi\frac{M-1}{2} + h\frac{\sigma^2}{2}\xi^2\frac{(M-1)(2M-1)}{6M}$, because $\sum_{j=1}^{M}(M-j) = \frac{M(M-1)}{2}$ and $\sum_{j=1}^{M}(M-j)^2 = \frac{M(M-1)(2M-1)}{6}$. Thus

$F(S_t,t) = \frac{Ke^{-r(T-t)}}{2\pi i}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\exp[i\xi(\ln\frac{S_t}{K} + \frac{1}{2}(r-\frac{\sigma^2}{2})(T-T')) - (T-T')\frac{\sigma^2}{2}\xi^2\frac{2M-1}{6M}]$
$\cdot(\frac{1}{\xi+i} - \frac{1}{\xi})d\xi$

for any $\omega > 0$. Splitting the integral into two integrals and changing variables, one gets:

$F(S_t,t) = \frac{S_te^{-\beta}}{2\pi i}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\exp[i\eta D^+ - \frac{\sigma^2}{2}\eta^2]\frac{1}{\eta}d\eta - \frac{Ke^{-r(T-t)}}{2\pi i}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\exp[i\eta D^- - \frac{\sigma^2}{2}\eta^2]\frac{1}{\eta}d\eta$

where $D^- = [\ln\frac{S_t}{K} + \frac{1}{2}(r - \frac{\sigma^2}{2})(T-T')]/(\sigma\sqrt{T-T'}\sqrt{\frac{2M-1}{6M}})$,
$D^+ = D^- + \sigma\sqrt{T-T'}\sqrt{\frac{2M-1}{6M}}$ and $\beta = r(T-t) + (r + \sigma^2(\frac{1}{2} - \frac{2M-1}{6M}))\frac{T-T'}{2}$, which yields the known formula (see [29]) by application of Lemma 1.

Finally we point out that Proposition 1 straightforwardly applies to the more general flexible geometric Asian options, where the flexible geometric average is $\prod_{j=1,...,M} S_{T_j}^{\theta_j}$ with $\theta_j = \theta(j)/\sum_{j=1}^{M}\theta(j)$ and θ any non-negative function (see [29] for the Gaussian case). The following expression is obtained:

$$F(S_t,t) = -\frac{Ke^{-r(T_M-t)}}{2\pi}\int_{-\infty-iw\omega}^{+\infty-iw\omega}\frac{1}{\xi(\xi+i)}\exp[i\xi\ln\frac{S_t}{K} - \sum_{j=1}^{M}(T_j - T_{j-1})\psi(\xi\sum_{k=j}^{M}\theta_k)]d\xi$$

with $T_0 = t$, $\omega \in]1, -\lambda_-[$ $(]0, \lambda_+[)$ if $w = 1$ $(w = -1)$.

3) *Discrete lookback options.* Lookback options are path-dependent options whose payoff depends on the extremal price of the underlying asset over the life of the option. We focus on the realistic case of finite sampling lookback, that is, the asset price is monitored at particular dates. Moreover, we confine ourselves to fixed strikes options, that is, the payoff is the maximum difference between the optimal price and the strike price K which is determined at inception. According to [12] "perhaps, in exponential Lèvy model closed-form formulas are not, in general,

available for pricing these options". In the traditional Gaussian setting discrete lookback options have been priced in [20] exactly, and in [8] by an adjustment of the continuous case.

Let $T_1 < T_2 < ...T_M$ be the monitoring times and assume that the lookback period $[T_1, T_N]$ starts in the future, i.e. the current time $t < T_1$, and ends at expiration, i.e. $T_M = T$, the expiration date. However our method applies also to backward starting options. The payoff is $\max(S_{T_1}, ..., S_{T_M}, K) - K$ for the call and $K - \min(S_{T_1}, ..., S_{T_M}, K)$ for the put, that is, $\max(wS_{T_1}, ..., wS_{T_M}, wK) - wK$ for any option, where $w = \pm 1$. Following [20] one can write the payoff in our notation as follows:

$$\sum_{p=1}^{M} we^{X_p}[\mathbf{1}_{M-1}(wA^{(p)}\mathbf{X} > 0) - \mathbf{1}_M(w\widetilde{A}^{(p)}\mathbf{X} > wB^{(p)})] - wK(1 - \mathbf{1}_M(w\mathbf{X} < w\widehat{K}))$$

where

$A^{(p)}$ is an $(M-1) \times M$ matrix whose entries are $A^{(p)}_{ij} = \begin{cases} 1 & if \quad j = p \\ -1 & if \quad i = j \neq p \\ 0 & \quad\ otherwise \end{cases}$;

$\widetilde{A}^{(p)}$ is an $M \times M$ matrix whose entries are $\widetilde{A}^{(p)}_{ij} = \begin{cases} 1 & if \quad j = p, i \neq j \\ -1 & if \quad i = j \\ 0 & \quad\ otherwise \end{cases}$;

$B^{(p)}$ is an $M-$dimension vector whose entries are 0 with the exception of the p^{th} that is $-\ln K$;

$\widehat{K}$ is an $M-$dimension vector whose entries are all $-\ln K$.

Thus Proposition 1 yields the following expression for the current value $F(S_t, t)$ of the lookback option:

$$\begin{aligned}
(4.1)\qquad & \frac{w^M S_t e^{-r(T_M-t)}}{(2\pi i)^{M-1}} \sum_{p=1}^{M} \int_{-\infty-iw\omega_1}^{+\infty-iw\omega_1} ... \int_{-\infty-iw\omega_M}^{+\infty-iw\omega_M} \frac{1}{\prod\limits_{\substack{n=1,...N\\ n\neq p}} \xi_n} \exp[-\sum_{j=1}^{p}(T_j - T_{j-1}) \\
& \psi(\sum_{n<j}\xi_n - i)] - \sum_{j=p+1}^{M}(T_j - T_{j-1})\psi(-\sum_{n\geq j}\xi_n)] d\xi_1 ... d\xi_{p-1} d\xi_{p+1} ... d\xi_M \\
& -\frac{w^M S_t e^{-r(T_M-t)}}{(2\pi i)^{M-1}} \sum_{p=1}^{M} \int_{-\infty-iw\omega_1}^{+\infty-iw\omega_1} ... \int_{-\infty-iw\omega_M}^{+\infty-iw\omega_M} \frac{1}{\prod\limits_{\substack{n=1,...N\\ n\neq p}} \xi_n} \exp[-\sum_{j=1}^{p}(T_j - T_{j-1}) \\
& \psi(\sum_{n<j}\xi_n - i) - \sum_{j=p+1}^{M}(T_j - T_{j-1})\psi(-\sum_{n\geq j}\xi_n)] d\xi_1 ... d\xi_{p-1} d\xi_{p+1} ... d\xi_M \\
& -\frac{w^{M+1} S_t e^{-r(T_M-t)}}{(2\pi i)^{M}} \sum_{p=1}^{M} \int_{-\infty-iw\omega_1}^{+\infty-iw\omega_1} ... \int_{-\infty-iw\omega_M}^{+\infty-iw\omega_M} \frac{1}{\prod\limits_{n=1,...N} \xi_n} \exp[-i\xi_p \ln\frac{S_t}{K} \\
& -\sum_{j=1}^{p}(T_j - T_{j-1})\psi(\sum_{n<j}\xi_n - \xi_p - i) - \sum_{j=p+1}^{M}(T_j - T_{j-1})\psi(-\sum_{n\geq j}\xi_n)] d\xi_1 ... d\xi_M \\
& +\frac{w^{M+1} K e^{-r(T_M-t)}}{(2\pi i)^{M}} \int_{-\infty-iw\omega_1}^{+\infty-iw\omega_1} ... \int_{-\infty-iw\omega_M}^{+\infty-iw\omega_M} \frac{1}{\prod\limits_{n=1,...N} \xi_n} \exp[-i\sum_{n=1}^{M}\xi_n \ln\frac{S_t}{K} - \\
& \sum_{j=1}^{p}(T_j - T_{j-1})\psi(-\sum_{n=j}^{M}\xi_n)] d\xi_1 ... d\xi_M \ -wKe^{-r(T_M-t)}
\end{aligned}$$

with some positive ω_n such that such that $\sum_{n=1}^{M}\omega_n < \min(\lambda_+, -\lambda_- - 1)$.

Let us see how our general formula collapses to (21) of [20]. For any $p = 1, ..., M$ denote:
$q_p^{\pm} = [\ln\frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T_p - t)]/(\sigma\sqrt{T_p - t})$, $g_{pn}^{\pm} = (\frac{r}{\sigma} \pm \frac{\sigma}{2})\sqrt{T_p - T_n}$ for $n = 1, ..., p-1$, $h_{pn} = (\frac{\sigma}{2} - \frac{r}{\sigma})\sqrt{T_n - T_p}$ for $n = p+1, ..., M$. Split any integral in the first sum into two integrals, the former in $d\xi_1...d\xi_{p-1}$ and the latter in $d\xi_{p+1}...d\xi_M$, and then change to variables $\xi_n'\sigma\sqrt{T_p - T_n} = \xi_n$ $(\xi_n'\sigma\sqrt{T_n - T_p} = \xi_n$, respectively). Then the integrals in the first sum of (4.1) becomes the product of:

$$\frac{w^p S_t e^{-r(T_M - T_p)}}{(2\pi i)^{p-1}} \int_{-\infty - iw\omega_1}^{+\infty - iw\omega_1} ... \int_{-\infty - iw\omega_{p-1}}^{+\infty - iw\omega_{p-1}} \frac{1}{\prod\limits_{n=1,...p}\xi_n} \exp[\sum_{n=1}^{p-1}(i\xi_n g_{pn}^{+} - \frac{1}{2}\xi_n^2 - \sum_{m<n}\xi_n\xi_m\sqrt{\frac{T_p - T_n}{T_p - T_m}})]d\xi_1...d\xi_{p-1}$$

and

$$\frac{w^{M-p}}{(2\pi i)^{M-p}} \int_{-\infty - iw\omega_{p+1}}^{+\infty - iw\omega_{p+1}} ... \int_{-\infty - iw\omega_M}^{+\infty - iw\omega_M} \frac{1}{\prod\limits_{n=p+1,...M}\xi_n} \exp[\sum_{n=p+1}^{M}(i\xi_n h_{pn} - \frac{1}{2}\xi_n^2 - \sum_{m>n}\xi_n\xi_m\sqrt{\frac{T_n - T_p}{T_m - T_p}})]d\xi_{p+1}...d\xi_M.$$

Employing Lemma 1 each term in the first sum is transformed into:
$wS_t e^{-r(T_M - T_p)} N_{p-1}(wg_{p1}^{+}, ..., wg_{p,p-1}^{+}; \Delta_p) N_{M-p}(wh_{p,p+1}, ..., wh_{pM}; \Theta_p)$
where the correlation matrix Δ_p has typical element $\sqrt{\frac{T_p - T_n}{T_p - T_m}}$, $m < n$, and the correlation matrix Θ_p has typical element $\sqrt{\frac{T_n - T_p}{T_m - T_p}}$, $m > n$. A similar treatment on the terms in the second sum turns each term into:
$wS_t e^{-r(T_M - T_p)} N_p(wg_{p1}^{+}, ..., wg_{p,p-1}^{+}, -wq_p^{+}; \Delta_p^*) N_{M-p}(wh_{p,p+1}, ..., wh_{pM}; \Theta_p)$
where the correlation matrix Δ_p^* has typical element $\rho_{mn}^* = \sqrt{\frac{T_p - T_n}{T_p - T_m}}$ for $n > m$, $m = 1, ..., p-1$, $\rho_{pn}^* = \sqrt{\frac{T_p - T_n}{T_p - t}}$ for $n < p$, and $\rho_{nn}^* = 1$ for any n. Finally, the integrals in the third sum are turned into $wKe^{-r(T_M - t)} N_M(-wq_1^{-}, ..., -wq_M^{+}; \widehat{\Delta})$ where the correlation matrix has typical element $\widehat{\rho}_{hk} = \frac{\min(T_h - t, T_k - t)}{\sqrt{T_h - t}\sqrt{T_k - t}}$.

4) *Chooser options*
A chooser option gives its holder the right to decide at a prespecified time (choice date $= T_1$) before the maturity T whether he/she would like the option to be a call or a put option. As a straightforward application of Proposition 1 (with $A = I$) we give a valuation formula for simple chooser options, i.e. the call and the put have the same strike price K and maturity date T. Note that the decision whether the option is a call or a put depends on the value of:
$Max(C(S_{T_1}; K, T), P(S_{T_1}; K, T)) = C(S_{T_1}; K, T) + Max(Ke^{-r(T - T_1)} - S_{T_1}, 0)$.
In other words the choice is:
Call $\iff Ke^{-r(T - T_1)} < S_{T_1}$; Put $\iff Ke^{-r(T - T_1)} > S_{T_1}$.

The payoff can be expressed as:
$(S_T - K)\mathbf{1}_2(S_T > K, S_{T_1} > Ke^{-r(T - T_1)}) + (K - S_{T_1})\mathbf{1}_2(S_T < K, S_{T_1} < Ke^{-r(T - T_1)})$
Then, in view of Proposition 1, with $N = 1$ and $M = 2$, the price for the simple chooser option at time $t < T_1$ can be written in the form:

$$F(S_t, t) = A_1 - K * A_2 + K * A_3 - A_4$$

where the following choice are made in Proposition 1 for each term:
for A_1 : $\mathbf{K}= (Ke^{-r(T-T_1)}, K)$ $\gamma = (0,1)$ $(w_1, w_2) = (1,1)$
for A_2 : $\mathbf{K}= (Ke^{-r(T-T_1)}, K)$ $\gamma = (0,0)$ $(w_1, w_2) = (1,1)$
for A_3 : $\mathbf{K}= (Ke^{-r(T-T_1)}, K)$ $\gamma = (0,0)$ $(w_1, w_2) = (-1,-1)$
for A_4 : $\mathbf{K}= (Ke^{-r(T-T_1)}, K)$ $\gamma = (0,1)$ $(w_1, w_2) = (-1,-1)$.

Then
$A_1 = \frac{S_t e^{-r(T-t)}}{(2\pi i)^2} \int_{-\infty-i\omega_2}^{+\infty-i\omega_2} \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)+i\xi_2 \ln\frac{S_t}{K}-\Psi(t,\xi_1,\xi_2-i)} \frac{1}{\xi_1\xi_2} d\xi_1 d\xi_2$
In view of the residue theorem A_1 becomes:
$= A_4 + \frac{S_t e^{-r(T-t)}}{2\pi i} \int_{-\infty-i\omega_2}^{+\infty-i\omega_2} e^{i\xi_2 \ln\frac{S_t}{K}-(T-t)\psi(\xi_2-i)} \frac{1}{\xi_2} d\xi_2 +$
$+ \frac{S_t e^{-r(T-t)}}{2\pi i} \int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)-(T_1-t)\psi(\xi_1-i)} \frac{1}{\xi_1} d\xi_1$
On the other hand
$A_2 = \frac{e^{-r(T-t)}}{(2\pi i)^2} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} \int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)+i\xi_2 \ln\frac{S_t}{K}-\Psi(t,\xi_1,\xi_2)} \frac{1}{\xi_1\xi_2} d\xi_1 d\xi_2$
which under the residue theorem is transformed into:
$= A_3 + \frac{e^{-r(T-t)}}{2\pi i} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} e^{i\xi_2 \ln\frac{S_t}{K}-(T-t)\psi(\xi_2)} \frac{1}{\xi_2} d\xi_2 +$
$+ \frac{e^{-r(T-t)}}{2\pi i} \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)-(T_1-t)\psi(\xi_1)} \frac{1}{\xi_1} d\xi_1$

Thus the formula is simplified because the double integrals cancel out and the final expression is:
$F(S_t,t) = \frac{S_t e^{-r(T-t)}}{2\pi i} [\int_{-\infty-i\omega_2}^{+\infty-i\omega_2} e^{i\xi_2 \ln\frac{S_t}{K}-(T-t)\psi(\xi_2-i)} \frac{1}{\xi_2} d\xi_2 +$

$\int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)-(T_1-t)\psi(\xi_1-i)} \frac{1}{\xi_1} d\xi_1] -$

$\frac{Ke^{-r(T-t)}}{2\pi i} [\int_{-\infty+i\omega_2}^{+\infty+i\omega_2} e^{i\xi_2 \ln\frac{S_t}{K}-(T-t)\psi(\xi_2)} \frac{1}{\xi_2} d\xi_2 +$
$\int_{-\infty-i\omega_1}^{+\infty-i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K}+r(T-T_1)-(T_1-t)\psi(\xi_1)} \frac{1}{\xi_1} d\xi_1]$.

In the Gaussian case the formula becomes
$F(S_t,t) = S[N(d_T^+) - N(-d_{T,T_1}^+)] - Ke^{-r(T-t)}[N(d_T^-) - N(-d_{T,T_1}^-)]$
with $d_T^{\pm} = (\ln(S_t/K) + (r \pm \frac{\sigma^2}{2})(T-t))/(\sigma\sqrt{T-t})$
$d_{T,T_1}^+ = d_{T_1}^{\pm} + r(T-T_1)/(\sigma\sqrt{T_1-t})$

which is the price for a chooser option obtained by Rubinstein in 1991 (See also [29]).

5) Multicompound options. A notable example of application of our Proposition is the valuation of multicompound options (of order N), that is options whose underlying asset is a multicompound option of order $N-1$. A closed form expression for this kind of options was developed by [19] in the usual Gaussian setting. An extension to a more general Lévy environment has been provided in [3] for the compound call options of order 2. Here we give a valuation expression which holds for any order. Note that multicompound options have been introduced to face the problem of pricing defaultable coupon bonds ([19], [4]) ; however they have a wide range of applications to all opportunities having a sequential nature (see [1], [2] for applications in real option analysis).

Let $t < T_1 < T_2 < ...T_N$ and let $F_N(S_t, t; T_1, K_1, w_1; ...; T_N, K_N, w_N)$ denote the current value of a European N-fold compound option expiring at time T_1, with strike price K_1 and with as underlying asset a European $(N-1)$-fold compound option expiring at time T_2, with strike price K_2, ...until the final underlying asset,

a European option on a stock, with exercise date and price given by T_N and K_N. As usually $w_j = \pm 1$ represents the call/put attribute of each option. Assume that the stochastic process followed by the underlying stock is a RLPE. Suppose that S_j^* is the solution to $F_{N-j}(S_j^*, T_j; T_{j+1}, K_{j+1}, w_{j+1}; ...; T_N, K_N, w_N) = K_j$ for $j = 1, ..., N-1$. We will comment on existence and uniqueness of S_j^* later on. Then the payoff of the multicompound option can be written as:
$\prod_{n=1}^{N} w_n S_{T_N} \mathbf{1}_N(w_N S_{T_N} \geq w_N S_N^*, ..., w_1 S_{T_1} \geq w_1 S_1^*) - \sum_{j=1}^{N} \prod_{n=1}^{j} w_n K_j \mathbf{1}_j(w_j S_{T_j} \geq w_j S_j^*, ..., w_1 S_{T_1} \geq w_1 S_1^*)$
where $S_N^* = K_N$ therefore our method applies. The current value of each term is obtained by straightforward application of Proposition 1 and the resulting expression reads:

$$\frac{S_t e^{-r(T_M - t)}}{(2\pi i)^N} \int_{-\infty - iw_1\omega_1}^{+\infty - iw_1\omega_1} \cdots \int_{-\infty - iw_N\omega_N}^{+\infty - iw_N\omega_N} \frac{1}{\prod_{n=1,...N} \xi_n} \exp[i \sum_{n=1}^{N} \xi_n \ln \frac{S_t}{S_n^*}]$$
$$. \exp[-\Psi_N(t, \xi_1, .., \xi_N - i)] d\xi_1 ... d\xi_N -$$
$$\sum_{j=1}^{N} \frac{K_j e^{-r(T_j - t)}}{(2\pi i)^j} \int_{-\infty - iw_1\omega_1}^{+\infty - iw_1\omega_1} \cdots \int_{-\infty - iw_j\omega_j}^{+\infty - iw_j\omega_j} \frac{1}{\prod_{n=1,...j} \xi_n} \exp[i \sum_{n=1}^{j} \xi_n \ln \frac{S_t}{S_n^*}]$$
$$. \exp[-\Psi_j(t, \xi_1, .., \xi_j)] d\xi_1 ... d\xi_j$$

for some positive ω_n such that $\sum_{n=1}^{j} w_n \omega_n \in]-\lambda_+, -\lambda_- - 1[\ \forall j = 1, ..., N$, and where

$\Psi_j(t, \xi_1, .., \xi_j) = \sum_{k=1}^{j} (T_k - T_{k-1}) \psi(\sum_{n=1}^{j} \xi_n)$.

Note that, differentiating under the integral sign, one can prove that $\partial_{h_j} F_N = 0$ for $h_j = \frac{S_t}{S_j^*}$, $j = 1, ..., N$. Thus $\partial_{S_t} F_N$ is just the first integral of F_N divided by S_t. Therefore uniqueness of S_j^* is guaranteed for any j. Existence holds for the multicompound call options, while in the general case it holds only for suitable strike prices.

In the Gaussian case, if we change variables $\xi_j \sigma \sqrt{T_j - t} = \eta_j$, let ω_j be any positive number and denote $(\ln \frac{S_t}{S_j^*} + (r \pm \frac{\sigma^2}{2})(T_j - t))/(\sigma \sqrt{T_j - t})$ by $d_j^{\pm}$, and $\sqrt{\frac{T_j - t}{T_k - t}}$ by ρ_{jk} if $j < k$, then our formula collapses into:
$S_t \prod_{n=1}^{N} w_n N_N(w_N d_N^+, ..., w_1 d_1^+; \Xi_N) - \sum_{j=1}^{N} \prod_{n=1}^{j} w_n K_j N_j(w_j d_j^-, ..., w_1 d_1^+; \Xi_j)$
where Ξ_j is a $j \times j-$ matrix with typical elements $\rho_{jk} w_k w_j$ when $j < k$.

Remark. (Put-call parity for compound options). Note that, starting from our formula, one can also verify the put-call parity relationship for compound options as a nice exercise. Indeed

$$F_2(S_t, t; 1, K_1, T_1; w_2, K_2, T_2) =$$
$$= \frac{S_t e^{-r(T_2 - t)}}{(2\pi i)^2} \int_{-\infty - iw_2\omega_2}^{+\infty - iw_2\omega_2} \int_{-\infty - i\omega_1}^{+\infty - i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2 - i)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 -$$
$$\frac{K_2 e^{-r(T_2 - t)}}{(2\pi i)^2} \int_{-\infty - iw_2\omega_2}^{+\infty - iw_2\omega_2} \int_{-\infty - i\omega_1}^{+\infty - i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 -$$
$$\frac{K_1 e^{-r(T_1 - t)}}{2\pi} \int_{-\infty - i\omega_1}^{+\infty - i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} - (T_1 - t)\psi(\xi_1)} \frac{1}{i\xi_1} d\xi_1.$$

Let us shift the line of integration $\mathrm{Im}\, \xi_1 = \omega_1$ up. Since we cross the pole at $\xi_1 = 0$, the residue theorem gives:

$F_2(S_t,t;1,K_1,T_1;w_2,K_2,T_2) =$
$\frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2}\int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2}\int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln\frac{S_t}{S^*}+i\xi_2\ln\frac{S_t}{K_2}-\Psi(t,\xi_1,\xi_2-i)}\frac{1}{\xi_1\xi_2}d\xi_1 d\xi_2-$
$\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2}\int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2}\int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln\frac{S_t}{S^*}+i\xi_2\ln\frac{S_t}{K_2}-\Psi(t,\xi_1,\xi_2)}\frac{1}{\xi_1\xi_2}d\xi_1 d\xi_2-$
$\frac{K_1 e^{-r(T_1-t)}}{2\pi}\int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln\frac{S_t}{S^*}-(T_1-t)\psi(\xi_1)}\frac{1}{i\xi_1}d\xi_1+$

$+2\pi i[\frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2}\int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} e^{i\xi_2\ln\frac{S_t}{K_2}-\Psi(t,0,\xi_2-i)}\frac{1}{\xi_2}d\xi_2-$
$\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2}\int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} e^{i\xi_2\ln\frac{S_t}{K_2}-\Psi(t,0,\xi_2)}\frac{1}{\xi_2}d\xi_2-\ \frac{K_1 e^{-r(T_1-t)}}{2\pi i}e^{-(T_1-t)\psi(0)}] =$
$F_2(S_t,t;-1,K_1,T_1;w_2,K_2,T_2)+F_1(S_t,t;w_2,K_2,T_2)-K_1e^{-r(T_1-t)}.$

In summary, we have verified that the value of a call on an option equals the value of the corresponding put on the same option plus the value of the underlying option diminished by $K_1e^{-r(T_1-t)}$, where K_1 and T_1 are the strike price and the maturity date of the compound option, which is the put-call parity for compound options.

6) *Discrete barrier options.* Most analytical pricing formulas for barrier options assume continuous monitoring of the barrier, which corrisponds to some practical cases (e.g. in FX markets). However in practice the barrier might be monitored only at discrete points in time (e.g., at the close of the market). A discrete barrier option is either knocked in or knocked out if the price of the underlying asset is across the barrier at the time it is monitored. In the Gaussian case the pricing formulas have been studied by [8], [9], [16]; in the Lévy process models an interesting survey is presented in [22], where the novel method of [16] is also discussed. (See also [24] for a numerical approach). In this subsection we derive a valuation formula for a discrete barrier option as a further straightforward application of Proposition 2. While there exists eight barrier options types, depending on the barrier knocking in or out, on the barrier being above or below the initial value of spot (up or down) and on the call/put attribute, we confine ourselves to a down-and-out call without rebate. The other cases can be treated similarly.

Let B denote the level of the barrier and suppose that the underlying asset is monitored at times T_j, $j=1,...,M-1$ before the option expiry T_M. The payoff is $(S_{T_M}-K)\mathbf{1}_M(S_{T_j}>B, j=1,...,M-1; S_{T_M}\geq K)$. Therefore Proposition 2 with $N=M$ and $A=I$, the $M\times M$ identity matrix, yields:

$$F(S_t,t)=\frac{Ke^{-r(T_M-t)}}{(2\pi i)^M}\int_{-\infty-i\omega_N}^{+\infty-i\omega_N}...\int_{-\infty-i\omega_1}^{+\infty-i\omega_1}\frac{1}{(\xi_M+i)\prod\limits_{k=1,...M-1}\xi_j}$$
$$.\exp[i\sum_{j=1}^{M-1}\xi_j\ln\frac{S_t}{B}+i\xi_M\ln\frac{S_t}{K}-\sum_{j=1}^{M}(T_j-T_{j-1})\psi(\sum_{k=j}^{M}\xi_k)]d\xi_1..d\xi_M$$

with $T_0=t$, $\omega_M\in]1,-\lambda_-[$, $\omega_j>0$ and $\sum\limits_{j=1}^{M}\omega_j<-\lambda_-$.

5. Conclusion

This work introduces a comprehensive option pricing formula for a very general family of payoffs, which includes many market-relevant option payoffs as special cases. The proof is based on Fourier methods and on the theory of pseudo differential operators that have been successfully applied in the literature for option pricing

in Lévy models. However, one does not need to be equipped with such mathematical sophistication in order to apply the main formula and thus the result may be of interest also for practitioners. The unifying formula we provide encompasses many existing option pricing expressions and is a powerful tool for generating new valuation expressions without effort. We have chosen to focus on discretely monitored options, as these have received little attention in the literature, despite their popularity in the trading practice. Each example is complemented with its Gaussian counterpart and thus, while introducing new formulas in the general Lévy setting, the paper may also serve as a review on discretely monitored options in the traditional Black-Scholes setting. Finally we stress that the analytical method based on pseudo differential operators and integration in the complex plane generates a new numerical method (integration-along-cut method) which often performs better than the Fast Fourier Transform (see [7]). Therefore numerical computation will take advantage of our explicit solutions from many a point of view.

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